EFFECT OF JOULE HEAT ON THE DISTRIBUTION OF ELECTRIC CURRENT IN A CYLINDRICAL CONDUCTOR

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Analyzed are the thermal conditions in a cylindrical conductor whose electrical conductivity is temperature-dependent. It is shown that a sufficiently large current is crowded toward the conductor surface with a change in the heat balance at the surface. The behavior of cylindrical semiconductors under analogous conditions is also examined.

In this study the authors consider the thermal conditions in a cylindrical conductor heated electrically with direct current, the case where the electrical resistivity of the material is a function of the temperature and when the rate of heat transfer at the surface is high.

The equation of heat conduction for an infinitely long cylinder with volume heat sources is

$$\nabla[\lambda(T)\nabla T]+W=0. \tag{1}$$

We calculate W:

$$W=\frac{kV_0^2}{\rho(T)}.$$

Here $V_0 = I_\Sigma/\sigma_\Sigma$, with $\sigma_\Sigma = 1/R_\Sigma = \int\limits_0^{r_0}\frac{2\pi r dr}{\rho(T)}$. Equation (1) can be written as

$$\frac{1}{r}\cdot\frac{d}{dr}\left[\lambda(T)\,r\,\frac{dT}{dr}\right]+\frac{kI_\Sigma^2}{\rho(T)\left[\int\limits_0^{r_0}\frac{2\pi r dr}{\rho(T)}\right]^2}=0.$$

With the Kirchhoff integral analog $\Phi = \int\limits_0^T \lambda(T)dt$ we reduce the last equation to

$$\frac{d^2\Phi}{dr^2}+\frac{1}{r}\cdot\frac{d\Phi}{dr}+\frac{kI_\Sigma^2}{\rho(\Phi)\left[\int\limits_0^{r_0}\frac{2\pi r dr}{\rho(\Phi)}\right]^2}=0. \tag{2}$$

The integrodifferential equation (2) contains an empirical function $\rho(\Phi)$. Over a certain interval $[0, r_0]$ of $\Phi(r)$ one may approximate this relation in several different ways. With an exponential relation this equation will be integrated most easily.

Let

$$\rho(\Phi)=\rho(\Phi_W)e^{a(\Phi-\Phi_W)},$$

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$$\frac{d^2\Phi}{dr^2}+\frac{1}{r}\cdot\frac{d\Phi}{dr}+\frac{\beta}{e^{\alpha(\Phi-\Phi_W)}}=0,$$

where

$$\beta=\frac{kI_\Sigma^2}{\rho(\Phi_W)\left[\int\limits_0^{r_0}\frac{2\pi r dr}{\rho(\Phi)}\right]^2}.$$

We now introduce the dimensionless variables

$$Y=\alpha(\Phi-\Phi_W),\ X=r/r_0,\ Y''+\frac{1}{X}Y'+\gamma e^{-Y}=0, \tag{3}$$

where $\gamma = \beta r_0^2\alpha$ is the dimensionless closure parameter. Integrating Eq. (3) yields

$$Y=\ln\frac{\gamma(e^{cc_1}X^c-1)^2X^2}{2c^2X^ce^{cc_1}}. \tag{4}$$

In order to determine the constants c and c_1, we use the boundary condition of the first kind T_W = const and the condition that the temperature at the cylinder axis is bounded:

$$\begin{aligned}&1)\ \text{for}\ X=1\quad Y=0,\\&2)\ \text{for}\ X=0\quad Y'=0.\end{aligned} \tag{4*}$$

These boundary conditions yield c = 2 and

$$c_2=\frac{1}{e^{2c_1}}=\frac{4+\gamma}{\gamma}\pm\frac{2\sqrt{2}}{\gamma}\sqrt{2+\gamma}.$$

With c and c_2 inserted into Eq. (4), we obtain

$$Y=\ln\frac{\gamma\left(X^2-\frac{4+\gamma}{\gamma}\mp\frac{2\sqrt{2}}{\gamma}\sqrt{2+\gamma}\right)^2}{8\left(\frac{4+\gamma}{\gamma}\pm\frac{2\sqrt{2}}{\gamma}\sqrt{2+\gamma}\right)}. \tag{5}$$

This is the solution to Eq. (3) with the boundary conditions (4*). The uniformity parameter can be determined after the integral $\int\limits_0^{r_0}\frac{2\pi r dr}{\rho(\Phi)}$ has been evaluated. The expression for γ becomes then

$$\gamma=\frac{kI_\Sigma^2\rho(\Phi_W)(4\pm2\sqrt{2}\sqrt{2+\gamma})^2\alpha}{64\pi^2r_0^2}.$$

Letting $[kI_\Sigma^2\alpha\rho(\Phi_W)]/64\pi^2r_0^2=\delta$, we rewrite the last equation in critical form:

$$\delta=\frac{\gamma}{(4\pm2\sqrt{2}\sqrt{2+\gamma})^2}. \tag{6}$$

Knowing the current I_Σ and the surface temperature, we can now determine δ and γ, and in these terms the most important process characteristics.

The rate of heat transfer at the surface will be determined in the following manner.

The rate of heat generation in the cylinder per unit length is $Q = \int_V W dV$. After evaluating the integral, we have

$$Q=\frac{4\pi\gamma}{(4\pm2\sqrt{2}\sqrt{2+\gamma})\alpha}.$$

Since $Q = 2\pi r_0$, hence

$$\frac{\gamma}{4\pm2\sqrt{2}\sqrt{2+\gamma}}=\frac{\alpha r_0 q}{2},\ q=\frac{2h}{\alpha r_0}.$$

Here

$$h = \frac{\gamma}{4 \pm 2\sqrt{2}\sqrt{2+\gamma}}. \tag{7}$$

Essential Process Characteristics

1. Temperature T_{max} or Φ_{max} at the axis:

$$\Phi_{max} = \Phi_W + \frac{1}{\alpha} \ln \frac{4+\gamma \pm 2\sqrt{2}\sqrt{2+\gamma}}{8}.$$

The formula for c_2 must be used here with the + sign, because otherwise $\Phi_{max} < \Phi_W$ for real values of γ and this would contradict the condition of heat transfer,

$$\Phi_{max} = \Phi_W + \frac{1}{\alpha} \ln \frac{4+\gamma + 2\sqrt{2}\sqrt{2+\gamma}}{8}. \tag{8}$$

2. Current crowding:

$$\frac{j(1)}{j(0)} = \frac{\rho(\Phi_{max})}{\rho(\Phi_W)} = \frac{4+\gamma+2\sqrt{2}\sqrt{2+\gamma}}{8} \tag{9}$$

3. Resistance of cylinder per unit length:

$$R_\Sigma = \frac{\rho(\Phi_W)(4 \pm 2\sqrt{2}\sqrt{2+\gamma})}{8\pi r_0^2}. \tag{10}$$

Calculations can be made with the aid of graphs shown in Fig. 1a, b. The curves in Fig. 1b correspond to materials with a positive temperature coefficient of resistance, the curves in Fig. 1a correspond to a negative temperature coefficient. The coefficient α as a function of the surface temperature is shown in Fig. 2 for various metals.

With known geometrical and thermophysical characteristics, one assumes the current I_Σ^2 and the surface temperature, whereupon the rate of heat transfer at the surface is found from (7), γ is found from (6), and the process characteristics are found from (8)-(10).

On the basis of the solution obtained here, one can draw the following conclusions:

1. In cylindrical metal or alloy conductors with a positive temperature coefficient of resistance the current which heats the conductor is crowded toward the surface and, as a result, the radial temperature gradient grad T decreases somewhat. The crowding factor is a function of I_Σ^2, r_0, α_T, T_W, λ, and other variables. For pure metals α_T is of the order of 0.5%/deg and λ varies from 360 kcal/m·h·deg for copper to 8 kcal/m·h·deg for bismuth [1]. Therefore, the crowding factor may be up to 5% for filaments in heater lamps or cathodes in electron tubes but 50% or higher for bismuth conductors.

2. If $\delta = 1/8$, then the closure coefficient γ tends toward infinity. This means a breakdown of the conductor, which now carries current only along the surface. In practice, however, melting of the conductor material at some finite value of γ will prevent this condition from occurring.

3. In cylindrical conductors of material with a negative temperature coefficient of resistance α_T (metal oxides, PbSe alloy, etc. [3]) the current is crowded inward and this results in a higher radial temperature gradient grad T, i.e., in a larger crowding factor. Under certain conditions our solution bcomes imaginary and, therefore, absurd. This happens when $\gamma < -2$. The process in a cylinder with $\gamma < -2$ requires further study. Since the absolute value of α_T is much higher for metal oxides than for metals, while λ is much lower and decreases further with rising temperature [3, 4, 10-13], hence $\gamma = -2$ is certainly attainable and this value of γ corresponds to a crowding factor $j(0)/j(1) = 4$. Thus, for example, the thermistor resistance can change by a factor of a few hundred [10], i.e., the crowding factor can be of the same order of magnitude under certain conditions of electric loading and heat transfer.

The crowding factor for modern KMT-1 thermistors is as high as 30-50% at the maximum power dissipation in air [3, 4].

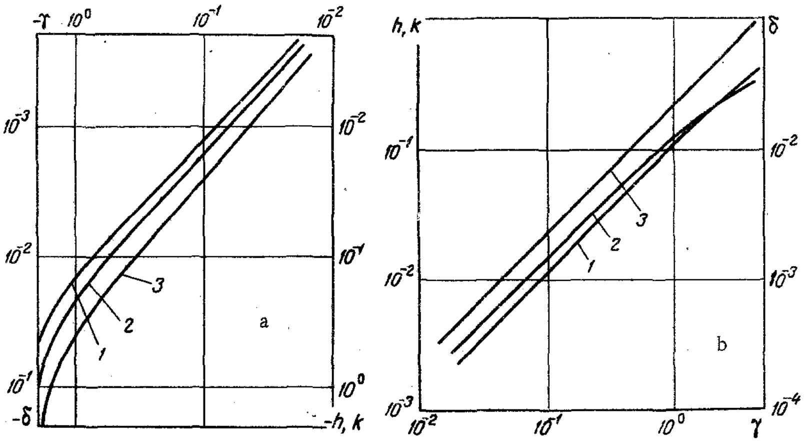

Fig. 1. a) Negative temperature coefficients and b) positive temperature coefficients: 1) $h(\gamma)$; 2) $\delta(\gamma)$; 3) $k(\gamma) = [j(1)-1]/j(0)$.

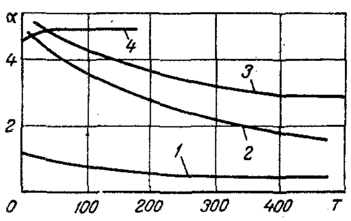

Fig. 2. Coefficient α (m · h/kcal) as a function of the wall temperature T (°C): 1) copper $10^5\alpha$; 2) platinum $10^5\alpha$; 3) tungsten 1800-2800°C, $10^6\alpha$; 4) bismuth $10^4\alpha$.

Dul'nev [5] has recognized the variation of electrical conductivity σ along a cylinder radius, but he disregarded this variation and assumed there σ = const(r) so as not to make the solution of the problem too unwieldy. It is of considerable practical interest, in solving various application problems on the subject of heat transfer, to compare the mean volume temperature calculated with the assumption of λ = const, W = const, or $\lambda = \lambda(T)$, $W = W(T)$ and boundary conditions of the first kind.

Example 1.

$$T = T_0\left[1 + \frac{\gamma^*}{2}(1 - X^2)\right],$$

where $\gamma^* = kI^2\rho_0/r_0^2\pi^2\lambda T_0$ is the closure parameter. The mean-over-the-section temperature of the conductor is

$$T_{\text{mean}} = \frac{1}{\pi r_0^2}\int_0^{r_0} 2\pi r T\,dr,$$

or

$$T_{\text{mean}} = T_0\left[1 + \frac{\gamma^*}{4}\right].$$

Considering that λ = const and $\Phi = \int_0^T \lambda dT = \lambda T$, one can write

$$\Phi_{\text{mean}} = \Phi_0\left[1 + \frac{\gamma^*}{4}\right]. \qquad (11)$$

Example 2.

$$\Phi_{\text{mean}} = \frac{1}{\pi r_0^2}\int_0^{r_0} 2\pi r\Phi\,dr \quad \text{or} \quad \Phi_{\text{mean}} = \int_0^1 \Phi\,d(X^2),$$

and, considering that

$$\Phi = \Phi_0 + \frac{1}{\alpha}\ln\frac{\gamma(X^2 - \nu)}{8\nu},$$

where

$$\nu = \frac{4+\gamma}{\gamma} + \frac{2\sqrt{2}}{\gamma}\sqrt{2+\gamma}, \tag{12}$$

we obtain

$$\Phi_{\text{mean}} = \Phi_0 + \frac{2}{\alpha}\left[2\nu \ln \frac{\nu}{\nu-1} - 1\right].$$

It can be shown that, as the electric loading is decreased down to zero, expressions (11) and (12) converge to the same result: Φ_0. The mean temperatures calculated by these two formulas can be compared only in specific cases.

A preliminary estimate has shown that Φ_{mean} calculated according to (12) is somewhat smaller than Φ_{mean} calculated according to (11) and that the difference increases with increasing γ and α.

NOTATION

- λ is the thermal conductivity;
- T is the temperature;
- ∇ is the operator in cylindrical coordinates;
- W is the rate of Joule heat generated in cylinder volume per unit length;
- V_0 is the voltage drop along cylinder, per unit length;
- $\rho(T)$ is the temperature-dependent electrical resistivity of the material;
- I_Σ is the total electric current across cylinder section;
- Φ_W is the Kirchhoff analog;
- α is the coefficient of approximation for the resistivity function;
- σ_Σ is the total electrical conductivity of cylinder, per unit length;
- q is the thermal flux density.

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